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STRESS SOLUTION DETERMINATION FOR HIGH ORDER PLATE THEORY

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Abstract—The high order theory of plate deformation developed in Ref.[1] and [2] is further examined herein. Specifically, stress solutions are given and evaluated against exact elasticity solutions under stringent short wave length load conditions. By the first method the stresses are evaluated directly from the resulting displacement solutions. In a more refined procedure, the transverse shear stresses and the transverse normal stress are evaluated by an alternate equilibrium method. The latter procedure is shown to be more accurate than the former.

INTRODUCTION

It has long been recognized that classical plate theory must be modified to treat certain high order effects. The first comprehensive generalization of the classical theory was that given by Reissner[3]. Since Reissner's work, there have been a great many further generalizations beyond the classical theory assumptions, with perhaps the highest order theory to date being that given by Lo, Christensen and Wu[1] and [2]. Preliminary steps were taken in Ref.[1] and [2] to assess the accuracy of the theory. In this paper this important subject is examined in greater detail.

The theory developed in Ref.[1] and [2] is based upon an assumed displacement field of the type

$$
\begin{aligned}
u &= u^{\circ}(x, y) + z\psi_x(x, y) + z^2\zeta_x(x, y) + z^3\phi_x(x, y) \\
v &= v^{\circ}(x, y) + z\psi_y(x, y) + z^2\zeta_y(x, y) + z^3\phi_y(x, y) \\
w &= w^{\circ}(x, y) + z\psi_z(x, y) + z^2\zeta_z(x, y)
\end{aligned}
\tag{1}
$$

where u and v are the in-plane displacement components, w the out of plane or transverse component, z the normal coordinate, and the remaining functions in (1) depend upon the in-plane coordinates x and y. The governing theory, based upon the principle of stationary potential energy, resulted in eleven second order partial differential equations to determine the eleven functions in (1). It appears that an approach of this type is the logical way to proceed if one wishes to determine only the displacements. It is less clear that this approach is the most expeditious method if one seeks to determine stresses. Now the comparisons with exact elasticity results given in Ref.[1] and [2] were only for the in-plane stress components, the transverse shear stresses and the transverse normal stress were not evaluated. Therefore the more complete stress information to be given here will help to answer the question of the general accuracy of the theory. Before proceeding with this however, it is useful to consider the three theoretical approaches to plate and shell development, and some advantages and disadvantages of each.

The first and most obvious approach to deriving an approximate plate theory utilizes assumptions upon the forms of displacements, as in (1). The governing differential equations could then be derived either by a direct method as in the case of classical plate theory, or by the use of the principle of stationary potential energy as in Ref.[1]. Equilibrium is violated by this approach, that is to say, the equilibrium equations are only approximately satisfied through weighted averages. The second possible approach is the direct reversal of that just described. Stress expansions in z are assumed and the governing differential equations are derived either by a direct approach or by the use of the complementary energy principle. Typical examples of this approach include the work of Ref.[4]. In this method the equilibrium equations are satisfied, the stress–strain relations are satisfied, but the compatibility of displacements is violated. In the third approach, assumptions are made upon both the stress states and the displacement forms whereby both equilibrium and compatibility conditions are satisfied. However the stress strain relations are violated. Reissner's plate theory is the most common form of this type [3].

It is the stress state which usually is the item of interest in most problems. Accordingly it might seem to be most rational to use either of the latter two stress type theories, but not the theory which depends exclusively upon displacement assumptions. However, such reasoning involves one serious problem, namely it would eliminate the extension of the theory to model laminate behavior. This consequence is because of the fact that in laminates the in-plane stress components are discontinuous and it becomes an exceedingly complex matter to construct a high order theory which must inherently account for discontinuous stresses. However, even in laminates the displacements, of course, are continuous, and since a major impetus in constructing new high order theories is for use with laminates, it is herein considered necessary to proceed with the displacement theory of Ref.[1]. The displacement theory of Ref.[1] in fact has been extended to model laminates by Lo, Christensen and Wu[2].

This status of affairs still leaves us with some uncertainty. Is the displacement type theory of Ref.[1] and [2] the best means by which to deduce the stress state under conditions where high order effects are of importance? The doubt arises because the equilibrium conditions are violated by the theory whereas stresses possess a one to one correspondence to the equilibrium conditions. However, there is one possible means by which the accuracy of the stresses obtained by this displacement type theory can be improved. The possible procedure is as follows. Use the high order theory based upon (1) to deduce the in-plane stress components, σ_x, σ_y, and τ_{xy}. Then insert these stress solutions into the equilibrium equations and solve for the out of plane/transverse stress components τ_{yz}, τ_{zx} and σ_z by integration. This procedure clearly results in a stress solution which satisfies equilibrium exactly. The procedure is suggested by the classical theory approach, which does not directly provide a solution for the transverse stress components, and they have to be found by the method described above.

Thus, the stresses implicit in the high order displacement type theory of Ref.[1] and [2] will be determined by two separate means. First the in-plane and transverse stresses will be found directly from the displacement solution through the use of the strain-displacement and stress strain relations. By the second method the in-plane stresses will be found by the method just described and they will then be used to determine the transverse stresses by integrating the equations of equilibrium. These two alternate methods of deducing stress will be compared and tested against exact elasticity solutions in the case of homogeneous plates. Then the second, more refined, procedure will be applied to laminates.

THEORETICAL CONSIDERATIONS

The assumed displacement fields are given by relations (1). The polynomial expansion for w is truncated at one order lower than the expansion for u and v such that the contributions to the transverse shear strains from u and v are of the same order in z as that from the terms in w. The strain-displacement relations of the linear theory of elasticity are

$$\begin{aligned} \epsilon_x &= u^\circ_{,x} + z\psi_{x,x} + z^2\zeta_{x,x} + z^3\phi_{x,x} \\ \epsilon_y &= v^\circ_{,y} + z\psi_{y,y} + z^2\zeta_{y,y} + z^3\phi_{y,y} \\ \epsilon_z &= \psi_z + 2z\zeta_z \end{aligned} \tag{2}$$

and

$$\begin{aligned} \gamma_{xy} &= \gamma^\circ_{xy} + z\Gamma_{xy} + z^2\beta_{xy} + z^3K_{xy} \\ \gamma_{xz} &= \gamma^\circ_{xy} + z\Gamma_{xz} + z^2\beta_{xz} \\ \gamma_{yz} &= \gamma^\circ_{yz} + z\Gamma_{yz} + z^2\beta_{yz} \end{aligned} \tag{3}$$

with

$$
\begin{aligned}
&\gamma^{\circ}_{xy}=u^{\circ}_{,y}+v_x; \quad \gamma^{\circ}_{xz}=\psi_x+w^{\circ}_{,x}; \quad \gamma^{\circ}_{yz}=\psi_y+w^{\circ}_{,y}\\
&\Gamma_{xy}=\psi_{x,y}+\psi_{y,x}; \quad \Gamma_{xz}=2\zeta_x+\psi_{z,x}; \quad \Gamma_{yz}=2\zeta_y+\psi_{z,y}\\
&\beta_{xy}=\zeta_{x,y}+\zeta_{y,x}; \quad \beta_{xz}=3\phi_x+\zeta_{z,x}; \quad \beta_{yz}=3\phi_y+\zeta_{z,y}\\
&K_{xy}=\phi_{x,y}+\phi_{y,z}.
\end{aligned}
\tag{4}
$$

The stress–strain relations appropriate to an anisotropic material are given by

$$
\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xy}\\ \tau_{xy}\end{bmatrix}
=
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\
 & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\
 & & C_{33} & C_{34} & C_{35} & C_{36}\\
 & & & C_{44} & C_{45} & C_{46}\\
\text{Sym} & & & & C_{55} & C_{56}\\
 & & & & & C_{66}
\end{bmatrix}
\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}
\tag{5}
$$

where C_{ij}, $i, j = 1, 2, \ldots 6$ are the stiffness coefficients. The derivation of the governing equations for this higher order plate theory is given in Ref. [1] for homogeneous isotropic plates, and its extension to laminated plate conditions is given in Ref. [2]. The evaluation of the stresses will now be given by the two methods mentioned in the introduction.

STRESS EVALUATION, HOMOGENEOUS PLATE

An infinite homogeneous isotropic plate is subjected to sinusoidal loadings as in Ref. [1]. The prescribed surface tractions are

$$\sigma_z(h/2)=q_0\sin\left(\frac{\pi x}{L}\right); \quad \sigma_z(-h/2)=0 \tag{6}$$

and

$$\tau_{xz}(\pm h/2)=\tau_{yz}(\pm h/2)=0 \tag{7}$$

where h is the plate thickness.

The solutions to the plate problems are [1].

$$
\begin{aligned}
&u=[A_0+zA_1+z^2A_2+z^3A_3]\cos\left(\frac{\pi x}{L}\right)\\
&v=0\\
&w=[C_1+zC_2+z^2C_3]\sin\left(\frac{\pi x}{L}\right)
\end{aligned}
\tag{8}
$$

where the constants A_0, A_1, A_2, A_3 and C_1, C_2 and C_3 are obtained by satisfaction of the governing differential equations and boundary conditions. From (3)–(5) the transverse stress components appropriate to a homogeneous isotropic plate are

$$
\begin{aligned}
\sigma_z=\;&\lambda(u^{\circ}_{,x}+v^{\circ}_{,y})+(\lambda+2\mu)\psi_z\\
&+z[\lambda(\psi_{x,x}+(\psi_{y,y})+2(\lambda+2\mu)\zeta_z]\\
&+z^2\lambda(\zeta_{x,x}+\zeta_{y,y})+z^3\lambda(\phi_{x,x}+\phi_{y,y})
\end{aligned}
$$

and

$$
\begin{aligned}
&\tau_{xz}=\mu(\psi_x+w^{\circ}_{,x})+z\mu(2\zeta_x+\psi_{z,x})+z^2\mu(3\phi_x+\zeta_{z,x})\\
&\tau_{yz}=\mu(\psi_y+w^{\circ}_{,y})+z\mu(2\zeta_y+\psi_{z,y})+z^2\mu(3\phi_y+\zeta_{z,y}).
\end{aligned}
\tag{10}
$$

Substituting (8) into (9) and (10) gives the transverse stress components for the infinite plate

$$\sigma_z=\left\{-\lambda A_0\frac{\pi}{L}+(\lambda+2\mu)C_2+z\left[-\lambda A_1\frac{\pi}{L}+2(\lambda+2\mu)C_3\right]-z^2\lambda A_2\frac{\pi}{L}-z^3\lambda A_3\frac{\pi}{L}\right\}\sin\left(\frac{\pi x}{L}\right) \tag{11}$$

and

$$\tau_{xz}=\left\{\mu\left[A_1+C_1\frac{\pi}{L}\right]+z\mu\left[2A_2+C_2\frac{\pi}{L}\right]+z^2\mu\left[3A_3+C_3\frac{\pi}{L}\right]\right\}\cos\left(\frac{\pi x}{L}\right) \tag{12}$$

$$\tau_{yz}=0.$$

These stresses are of course those deduced directly from the theory, and they will be displayed graphically for particular values of h/L, i.e. the ratio of the thickness of the plate to the characteristic length of the loading pattern. First of all, however, these same stress components will be evaluated by the alternate method mentioned in the introduction.

The in-plane stress components for the problem under consideration are given in Ref.[1]. These in-plane stresses are substituted into the equilibrium equations

$$\sigma_{ij,j}=0 \tag{13}$$

to yield, after integration, the out of plane/transverse stress components:

$$\begin{aligned}\sigma_z=\Bigg\{\frac{q_0}{2}&+\frac{1}{2}\left[z^2-\frac{h^2}{4}\right]\left[(\lambda+2\mu)A_0\frac{\pi^3}{L^3}-\lambda C_2\frac{\pi^2}{L^2}\right]\\&+\frac{1}{12}\left[z^4-\frac{h^4}{16}\right](\lambda+2\mu)A_2\frac{\pi^3}{L^3}\\&+\frac{z}{2}\left[\frac{z^2}{3}-\frac{h^2}{4}\right]\left[(\lambda+2\mu)A_1\frac{\pi^3}{L^3}-2\lambda C_3\frac{\pi^2}{L^2}\right]\\&+\frac{z}{4}\left[\frac{z^4}{5}-\frac{h^4}{16}\right](\lambda+2\mu)A_3\frac{\pi^3}{L^3}\Bigg\}\sin\left(\frac{\pi x}{L}\right)\end{aligned} \tag{14}$$

and

$$\begin{aligned}\tau_{xz}=\Bigg\{z&\left[(\lambda+2\mu)Z_0\frac{\pi^2}{L^2}-\lambda C_2\frac{\pi}{L}\right]\\&+\frac{1}{2}\left[z^2-\frac{h^2}{4}\right]\left[(\lambda+2\mu)A_1\frac{\pi^2}{L^2}-2\lambda C_3\frac{\pi}{L}\right]\\&+\frac{z^3}{3}(\lambda+2\mu)A_2\frac{\pi^2}{L^2}+\frac{1}{4}\left[z^4-\frac{h^4}{16}\right](\lambda+2\mu)A_3\frac{\pi^2}{L^2}\Bigg\}\cos\left(\frac{\pi x}{L}\right)\end{aligned} \tag{15}$$

where the top and bottom surface traction conditions (6) and (7) have been used to evaluate the constants of integration.

The corresponding exact elasticity solution for this problem is given in Ref.[5], among many other sources. Before comparing the two alternate means of deriving the stresses, first the corresponding results for a particular liminated plate will be stated.

LAMINATED PLATE

The laminate to be considered is that of symmetric cross-ply geometry with each lamina being orthotropic. Using relations (5), (8) and (13), the transverse stress components are found to be

$$\begin{aligned}\tau_{xz}^{(k)}=\Bigg\{-z&\left[-C_{11}^{(k)}A_0\frac{\pi^2}{L^2}+C_{13}^{(k)}C_2\frac{\pi}{L}\right]-\frac{z^2}{2}\left[-C_{11}^{(k)}A_1\frac{\pi^2}{L^2}+2C_{13}C_3\frac{\pi}{L}\right]\\&+\frac{z^3}{3}\left[C_{11}^{(k)}A_2\frac{\pi^2}{L^2}\right]+\frac{z^4}{4}\left[C_{11}^{(k)}A_3\frac{\pi^2}{L^2}\right]\Bigg\}\cos\left(\frac{\pi x}{L}\right)+f^{(k)}(x)\end{aligned} \tag{16}$$

and

$$\sigma_z^{(k)} = \left\{\frac{z^2}{2}\left[C_{11}^{(k)}A_0\frac{\pi^3}{L^3} - C_{13}^{(k)}C_2\frac{\pi^2}{L^2}\right] + \frac{z^3}{6}\left[C_{11}^{(k)}A_1\frac{\pi^3}{L^3} - 2C_{13}^{(k)}C_3\frac{\pi^2}{L^2}\right] + \frac{z^4}{12}C_{11}^{(k)}A_2\frac{\pi^3}{L^3} + \frac{z^5}{20}C_{11}^{(k)}A_3\frac{\pi^3}{L^3}\right\}\sin\left(\frac{\pi x}{L}\right) - zf^{(k)}(x)_{,x} + g^{(k)}(x) \tag{17}$$

where the index k refers to the kth layer and where $f^{(k)}$ and $g^{(k)}(x)$ are determined from the boundary and continuity conditions. These transverse stresses are those found by the method involved in using the in-plane stresses in the equations of equilibrium which are then integrated to find the three remaining transverse stress components.

Further consideration will be limited to the case of a three layer laminate arranged with the fiber directions designated by [0°/90°/0°] and with the laminae having equal thickness of $h/3$. The following expressions are obtained for $f^{(k)}(x)$ and $g^k(x)$.

$$f^{(1)}(x) = \left\{\frac{h}{2}\left[-C_{11}^{(1)}A_0\frac{\pi^2}{L^2} + C_{13}^{(1)}C_2\frac{\pi}{L}\right] + \frac{h^2}{8}\left[-C_{11}^{(1)}A_1\frac{\pi^2}{L^2} + 2C_{13}^{(1)}C_3\frac{\pi}{L}\right] - \frac{h^3}{24}C_{11}^{(1)}A_2\frac{\pi^2}{L^2} - \frac{h^4}{64}C_{11}^{(1)}A_3\frac{\pi^2}{L^2}\right\}\cos\left(\frac{\pi x}{L}\right) \tag{18}$$

$$f^{(2)}(x) = f^{(1)}(x) + \left\{\frac{h}{6}\left[-[C_{11}^{(2)} - C_{11}^{(1)}]A_0\frac{\pi^2}{L^2} + [C_{13}^{(2)} - C_{13}^{(1)}]C_2\frac{\pi}{L}\right] + \frac{h^2}{72}\left[-[C_{11}^{(2)} - C_{11}^{(1)}]A_1\frac{\pi^2}{L^2} + 2[C_{13}^{(2)} - C_{13}^{(1)}]C_3\frac{\pi}{L}\right] - \frac{h^3}{648}[C_{11}^{(2)} - C_{11}^{(1)}]A_2\frac{\pi^2}{L^2} - \frac{h^4}{5184}[C_{11}^{(2)} - C_{11}^{(1)}]A_3\frac{\pi^2}{L^2}\right\}\cos\left(\frac{\pi x}{L}\right) \tag{19}$$

$$f^{(3)}(x) = \left\{-\frac{h}{2}\left[-C_{11}^{(1)}A_0\frac{\pi^2}{L^2} + C_{13}^{(1)}C_2\frac{\pi}{L}\right] + \frac{h^2}{8}\left[-C_{11}^{(1)}A_1\frac{\pi^2}{L^2} + 2C_{13}^{(1)}C_3\frac{\pi}{L}\right] + \frac{h^3}{24}C_{11}^{(1)}A_2\frac{\pi^2}{L^2} - \frac{h^4}{64}C_{11}^{(1)}A_3\frac{\pi^2}{L^2}\right\}\cos\left(\frac{\pi x}{L}\right) \tag{20}$$

$$g^{(1)}(x) = q_0\sin\left(\frac{\pi x}{L}\right) + \frac{h}{L}f_x^{(1)}(x) - \left\{\frac{h^2}{8}\left[C_{11}^{(1)}A_0\frac{\pi^3}{L^3} - C_{13}^{(1)}C_2\frac{\pi^2}{L^2}\right] + \frac{h^3}{48}\left[C_{11}^{(1)}A_1\frac{\pi^3}{L^3} - 2C_{13}^{(1)}C_3\frac{\pi^2}{L^2}\right] + \frac{h^4}{192}C_{11}^{(1)}A_2\frac{\pi^3}{L^3} + \frac{h^5}{640}C_{11}^{(1)}A_3\frac{\pi^3}{L^3}\right\}\sin\left(\frac{\pi x}{L}\right) \tag{21}$$

$$g^{(2)}(x) = \frac{h}{6}\left[f_x^{(2)}(x) - f_x^{(1)}(x)\right] + g^{(1)}(x) + \left\{\frac{h^2}{72}\left[(C_{11}^{(1)} - C_{11}^{(2)})A_0\frac{\pi^3}{L^3} - (C_{13}^{(1)} - C_{13}^{(2)})C_2\frac{\pi^2}{L^2}\right] + \frac{h^3}{1296}\left[C_{11}^{(1)} - C_{11}^{(2)})A_1\frac{\pi^3}{L^3} - 2(C_{13}^{(1)} - C_{13}^{(2)})C_3\frac{\pi^2}{L^2}\right] + \frac{h^4}{15552}(C_{11}^{(1)} - C_{11}^{(2)})A_2\frac{\pi^3}{L^3} + \frac{h^5}{155520}(C_{11}^{(1)} - C_{11}^{(2)})A_3\frac{\pi^3}{L^3}\right\}\sin\left(\frac{\pi x}{L}\right)$$

and

$$g^{(3)}(x) = -\frac{h}{2}f_x^{(3)}(x) - \left\{\frac{h^2}{8}\left[C_{11}^{(1)}A_0\frac{\pi^3}{L^3} - C_{13}^{(1)}C_2\frac{\pi^2}{L^2}\right] - \frac{h^3}{48}\left[C_{11}^{(1)}A_1\frac{\pi^3}{L^3} - 2C_{13}^{(1)}C_3\frac{\pi^2}{L^2}\right] + \frac{h^4}{192}C_{11}^{(1)}A_2\frac{\pi^3}{L^3} - \frac{h^5}{640}C_{11}^{(1)}A_3\frac{\pi^3}{L^3}\right\}\sin\left(\frac{\pi x}{L}\right) \tag{23}$$

where the subscripts 1, 2 and 3 refer to the 0°, 90° and 0° layers, respectively.

The stress distributions will be displayed for lamina of the following properties which are typical of high modulus graphite/epoxy composites

$$\begin{aligned} &E_L = 25 \times 10^6 \text{ psi}; \quad E_T = 10^6 \text{ psi} \\ &G_{LT} = 0.5 \times 10^6 \text{ psi}; \quad G_{TT} = 0.2 \times 10^6 \text{ psi} \\ &\nu_{LT} = \nu_{TT} = 0.25. \end{aligned} \tag{24}$$

where L and T refer to the properties along and transverse to the fiber directions, respectively, and ν_{LT} is the Poisson's ratio measuring transverse strain under normal stress parallel to the fibers.

DISCUSSION

The transverse normal stress and transverse shear stress for the homogeneous isotropic plate case are given in Figs. 1–3. The difference in these figures is due to the variation in the ratio of h/L, i.e. the ratio of thickness to half wave length of the sinusoidal load. In Fig. 1, for $h/L = 1/4$, the normal stresses calculated by the two different means are compared with the exact solutions. For this small value of h/L, the transverse shear stresses calculated by the two

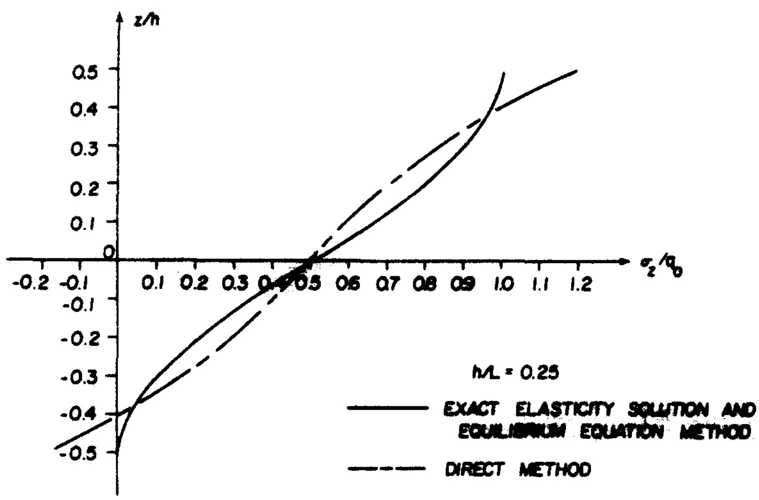

Fig. 1. Transverse normal stress distributions for a homogeneous isotropic plate at $h/L = 0.25$.

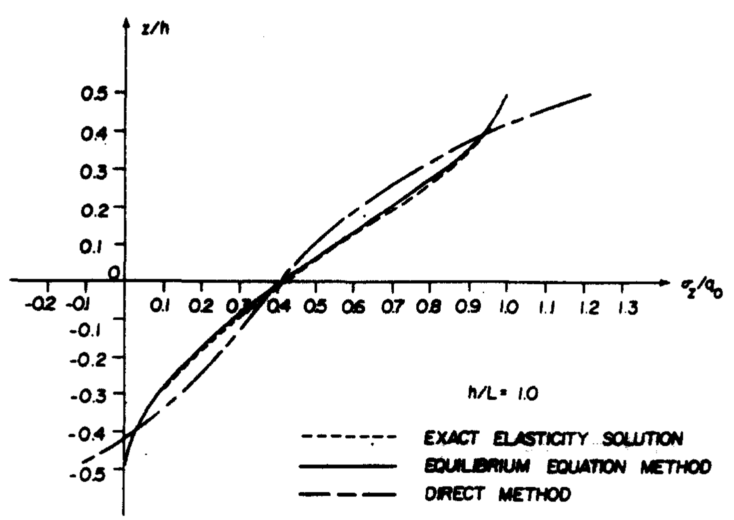

Fig. 2. Transverse normal stress distributions for a homogeneous isotropic plate at $h/L = 1.0$.

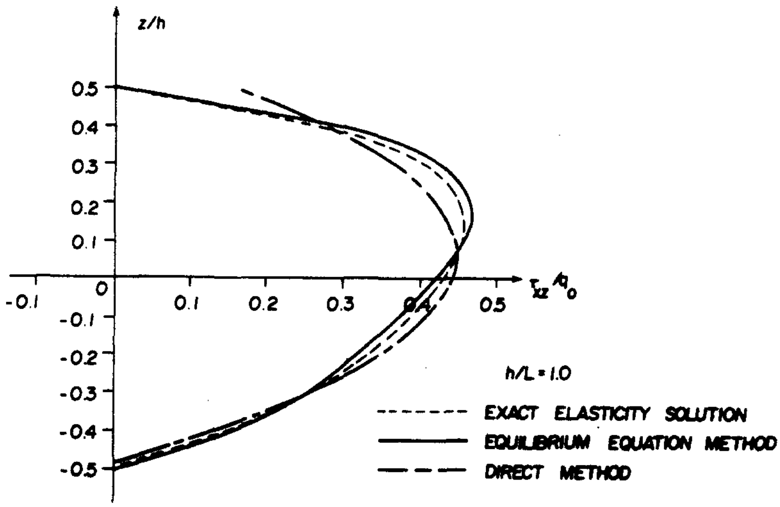

Fig. 3. Transverse shear stress distributions for a homogeneous isotropic plate at $h/L = 1.0$.

methods are so close to the exact solutions as to make them look identically equal graphically. Clearly the transverse normal stresses found by the integration of the equations of equilibrium are far more accurate than those obtained directly from the displacement solution through the strain displacement and the stress–strain relations. The results shown in these figures reveal that the transverse stresses found by equilibrium equation integration to be more accurate than those found directly from the displacement solution, with these results being under stringent short wave length load conditions. As discussed in Ref. [1] the maximum ratio of h/L for which the theory has reasonable validity is about $h/L = 1$ and the results shown here corroborate this conclusion.

It is of interest to note from Figs. 1–3 that the transverse stresses obtained directly from the displacement solution violate the top and bottom traction conditions. An examination of the derivation in Ref. [1] reveals this to be a consistent aspect of the method. Thus even though the tractions enter the theory as boundary conditions, this process actually occurs through an equilibrium weighting method, thus the theory does not provide exact satisfaction of these boundary conditions. Consider however, the alternate method of obtaining the transverse stresses from integrating the equilibrium equations utilizing the in-plane stresses found directly from the displacement solution. In this case the boundary tractions are automatically satisfied through the evaluation of the constants of integration. A similar situation exists in the case of

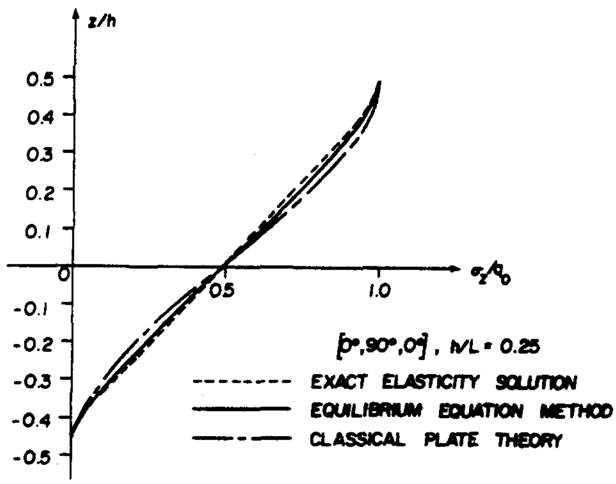

Fig. 4. Transverse normal stress distributions for a [0°, 90°, 0°] laminate at $h/L = 0.25$

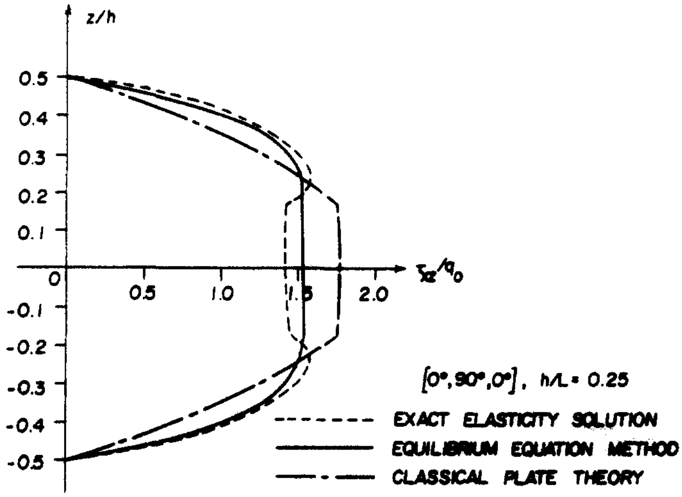

Fig. 5. Transverse shear stress distributions for a [0°, 90°, 0°] laminate at $h/L = 0.25$.

laminated plates. Transverse stresses evaluated directly from the displacement solution would in general not be continuous across the interfaces between lamina, however the transverse stresses found by the equilibrium method proposed herein provide continuous stress with exact satisfaction of top and bottom surface conditions.

The results for a three layer laminate are shown in Figs. 4 and 5, for the value of $h/L = 1/4$. The exact elasticity solution is taken from Pagano[6]. It is apparent that the case of the laminate provides much more strenuous conditions against which to test a plate theory than does homogeneous conditions. Nevertheless, as seen in these figures the equilibrium equation method of generating transverse stresses provides a reasonable approximation to the exact solution.

CONCLUSIONS

The present high order theory of plate deformation appears to provide reasonably accurate predictions of behavior under short wave length conditions. This conclusion is valid for both homogeneous plates and for laminates; also as shown by the results, laminates are much more demanding of high order effect representation than are homogeneous plates. In problems where displacements are the quantity of prime interest the present displacement type theory appears to provide a reasonable and high order effect solution. In problems where the stresses are the quantity sought it has been shown that the present theory still provides highly accurate stress information. It has been demonstrated that the best method for determining the stresses involves determining the in-plane stresses directly from the displacement solution and thence determining the transverse stresses through the integration of the equations of equilibrium, utilizing the in-plane solution therein. This method is of course applicable to a theory of any order not just the present high order theory. The success of the method was virtually assured by the fact that it is the only possible procedure for use at the level of the classical theory assumptions.

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